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A GENERALIZED SCHEME FOR CONSTRUCTING LYAPUNOV FUNCTIONS FROM FIRST INTEGRALS†

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A heuristic scheme is described for constructing Lyapunov v-functions, generalizing the classical method for constructing these functions from the first integrals of the equations of motion under investigation (or from the integrals of a comparison system). It is shown that the generalized scheme inherits a characteristic feature of the classical method: the Lyapunov functions are constructed as solutions of a certain completely integrable partial differential equation (or system of such equations). The form of this equation and its order are uniquely defined by a non-degenerate multi-parameter function $V(\mathbf{x}, \alpha) + \alpha_q$, $\mathbf{x} \in R^n$, $\alpha \in R^{q-1}$ (where α is a parameter vector), which generalizes the classical linear combination of integrals. Methods are described for representing v-functions, in the course of which the traditional methods (the method of Chetayev combinations of integrals and the construction of Lyapunov functions as a non-linear function of integrals) are augmented by geometrical constructions in which the v-functions are sought in the form of envelopes of certain subfamilies of the function $V(\mathbf{x}, \alpha) + \alpha_q$. The generalized scheme serves as a basis for deriving new, simple criteria for the asymptotic stability of the trivial solution in a transcendental problem of the stability of a system with two degrees of freedom in the critical case of two pairs of pure imaginary roots at 1 : 1 resonance (the case of simple elementary divisors).

1. THE CLASSICAL SCHEME FOR CONSTRUCTING V-FUNCTIONS

The most effective method for investigating stability problems is well-known to be the classical method of constructing v-functions from the first integrals of the equations of motion, which goes back to the work of Lagrange [1], Lyapunov [2] and Routh [3, 4]. The simplest case is that in which a positive-definite first integral is known. The stability of the system follows from Lyapunov's First Stability Theorem. In a more complex situation, which requires explicit construction of such an integral, one generally uses the method of Chetayev combinations of integrals, closely associated with which is a Routh–Lyapunov theorem [3, 4], which yields an explicit construction of v-functions of this type. This procedure has been used to investigate the stability of many conservative systems (e.g. [5–20]).

It is been shown that this approach is universal in the problem of constructing sufficient conditions for the stability of an equilibrium in Hamiltonian systems and reversible systems with analytic right-hand side: v-functions of the direct method which satisfy Lyapunov's First Stability Theorem are sign definite integrals of the system under investigation.

It is also well known that there is a wide range of non-conservative problems in which the v-functions belong to the space of first integrals of a certain auxiliary comparison system. In that case one usually applies the "energy approach", when the energy integral of the comparison system is considered as a Lyapunov function. The stability of steady motions of mechanical systems with dissipation has been investigated using combinations of integrals of a conservative comparison system [23–27] or using the generalized energy integral of the latter system [28].

The investigation of non-conservative problems has also been based on the construction of functional extensions of a combination of integrals of the comparison system. The basis for this modification of the classical method has been the study of gyroscopic systems with complete dissipation, using a linear combination of integrals plus quadratic terms [29, 30], or using a combination of the complete energy and extended cyclic integrals.‡ This device has also been used [31, 32] to investigate the stability of a gyroscope with dry friction.

The formal scheme of the classical approach is as follows. Suppose we are considering the stability of the trivial solution $\mathbf{x} = 0$ of a system of ordinary differential equations with fairly continuous right-hand sides

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}), \quad \mathbf{x} \in R^n \tag{1.1}$$

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‡MATROSOV V. M., Some problems of the stability of gyroscopic systems. Candidate dissertation, 05.11.03, Kazan, 1959.

To construct the required Lyapunov functions, we fix a comparison system

$$\dot{\mathbf{x}} = \mathbf{X}_0(\mathbf{x}) \tag{1.2}$$

If Eqs (1.1) are Hamiltonian or reversible system (1.2) must be identified with (1.1). Let us assume that independent integrals of Eqs (1.2) are known: $F_1(\mathbf{x}), \ldots, F_{m-1}(\mathbf{x})$, $m \leqslant n$. According to the idea of the approach, the v-functions are elements of the subset

$$M = \{F(\mathbf{x}):\ F = B(F_1, \ldots, F_{m-1}),\quad B \in C^r(R^{m-1}, R^1)\}$$

of the whole set of integrals of Eqs (1.2). The required Lyapunov functions will be sought as elements of the following form in the set M:

1) $\mathbf{v}(\mathbf{x})$ is a linear function of the integrals $F_j(\mathbf{x})$;

2) $\mathbf{v}(\mathbf{x})$ is a non-linear function of the integrals $F_j(\mathbf{x})$; one usually takes $\mathbf{v}(\mathbf{x})$ to be a Chetayev combination of integrals

$$\mathbf{v}(\mathbf{x}) = \sum_j \lambda_j F_j + \sum_j \mu_j F_j^2, \qquad \lambda_j, \mu_j = \text{const} \tag{1.3}$$

This heuristic scheme may be augmented by geometrical constructions, which are not directly applicable in applied problems but are necessary for the description of the classical approach to be complete.

There is a relation between the set M and partial differential equations (pde's) [33]. Let $W(\mathbf{x}, \mathbf{C}) = C_1F_1 + \ldots C_{m-1}F_{m-1}$, $\mathbf{C} = (C_1, \ldots, C_{m-1})$ be a non-degenerate combination of integrals. Then the set M coincides with the entire space of smooth solutions of a certain completely integrable system of first-order pde's

$$H_1(x, \mathbf{p}) = 0, \ldots, H_{1+n-m}(\mathbf{x}, \mathbf{p}) = 0, \quad \mathbf{p} \in R^n, \quad p_j = \frac{\partial z}{\partial x_j} \tag{1.4}$$

for which $W + C_m$ is a complete Lagrange integral. This means that the system of equations $p_j = \partial W/\partial x_j$ $(j = 1, \ldots, n)$ is uniquely solvable for the $m - 1$ quantities C_j, and, after eliminating them from the remaining $n - m + 1$ equations, it takes a form equivalent to (1.4). The dependence of H_j on $\mathbf{p}$ is linear, since $W + C_m$ depends linearly on C_j. In the limiting case $m = n$ (the comparison system (1.2) admits of a complete set of integrals), system (1.4) consists of a single linear equation

$$\sum_{j=1}^{n} X_{0j}(\mathbf{x}) p_j = 0 \tag{1.5}$$

It is also known [33] that any solution $z(\mathbf{x})$ of Eqs (1.4) may be obtained from the complete integral $W + C_m$ by Lagrange variation of the arbitrary constants C_j:

$$z(\mathbf{x}) = C_1(\mathbf{x})F_1(\mathbf{x}) + \ldots + C_{m-1}F_{m-1}(\mathbf{x}) + C_m(\mathbf{x})$$

where the vector of variations $(\mathbf{C}(\mathbf{x}), C_m(\mathbf{x}))$ is a solution of the Pfaff equation

$$\sum_{j=1}^{m-1} W'_{C_j} dC_j + dC_m = 0 \tag{1.6}$$

There is a geometrical interpretation of this method [33, 34]. Let π be an arbitrary regular l-surface in the space of the essential constants $C_1 \ldots, C_m (0 \leqslant l \leqslant m - 1)$, and let $(W + C_m)_\pi$ be the restriction of the family $(W + C_m)$ to the surface. Any solution $z(\mathbf{x})$ of Eqs (1.4) is the envelope (at least local) of some l-parameter family $(W + C_m)_\pi$ (if $l = 0$, the envelope is the function $(W + C_m)_\pi$ itself). Hence it follows that the solution space of Eqs (1.4) (that is, the set M) consists of the envelopes of all possible families $(W + C_m)_\pi$, with the subscript π running through the whole set of regular l-surfaces $(0 \leqslant l \leqslant m - 1)$ in the space of arbitrary constants C_j.

We now introduce further notation: M_l will denote the subspace of envelopes of all possible families $(W + C_m)_\pi$, when the subscript π runs through the whole set of regular l-surfaces in the space of arbitrary constants. Obviously

$$M = \bigcup_{l=0}^{m-1} M_l$$

The set M_0 is the simplest to construct, since its elements are functions of the form $W(\mathbf{x}, \mathbf{C}) + C_m$, $(\mathbf{C}, C_m) = \text{const}$. The other subspaces M_l are filled by envelopes. More detailed information on the structure of M in the case $n = m$ may be found in [35].

Thus, the classical scheme may be augmented. As the Lyapunov functions are solutions of the completely integrable system (1.4), they have one of the following representations:

3) $\mathbf{v}(\mathbf{x}) = C_1(\mathbf{x})F_1(\mathbf{x}) + \ldots + C_{m-1}F_{m-1}(\mathbf{x}) + C_m(\mathbf{x})$,
where $(C_1(\mathbf{x}), \ldots, C_m(\mathbf{x}))$ is a solution vector of Eq. (1.6);

4) $\mathbf{v}(\mathbf{x})$ is the envelope of some l-parameter family $(W + C_m)_\pi$.

Thus, the classical method of constructing Lyapunov functions from first integrals reduces to the procedures 1–4.

Note that the representation of Lyapunov functions as envelopes of families of functions is typical for stability problems. Indeed, according to Chetayev's method, v-functions should be sought in the form (1.3). If $\Sigma\mu_j^2 \neq 0$, then $\mathbf{v}(\mathbf{x})$ is not an element of M_0; therefore, $\mathbf{v}(\mathbf{x})$ belongs to the subset $M_1 \cup M_2 \cup \ldots \cup M_{m-1}^j$. But this means that $\mathbf{v}(\mathbf{x})$ is the envelope of some family $(W + C_m)_\pi$. The same conclusion holds when $\mathbf{v}$ is expressed as an arbitrary non-linear function of known integrals.

2. A GENERALIZED HEURISTIC SCHEME FOR CONSTRUCTING V-FUNCTIONS FROM FIRST INTEGRALS

Regardless of the effectiveness of the classical approach, its domain of application is generally limited to deriving the sufficient conditions for the stability of equilibrium in a conservative (or almost conservative) system. The point is that the majority of Lyapunov functions that satisfy stability theorems for a non-conservative system (as is the case for v-functions in the instability problem) are not integrals of the system in question, or of the comparison system.

We will describe a generalized heuristic method for constructing Lyapunov functions which is largely free from the above-mentioned shortcomings. The main idea is to assume that the desired Lyapunov functions belong to a certain function space generalizing the set M, which we will now describe.

Suppose we are given a smooth q-parameter function $V(\mathbf{x}, \alpha) + \alpha_q$, $q \geqslant m$, where $\alpha = (\alpha_1, \ldots, \alpha_{q-1})$ is a vector of arbitrary constants. Let us assume that this function satisfies a non-degeneracy condition

$$\operatorname{rank} V_{\mathbf{x},\alpha} = \min(n, q-1) \tag{2.1}$$

at every point of some range of values of the vectors $\mathbf{x}$ and α.

We will also assume that the complete integral of Eqs (1.4), that is, the function $W(\mathbf{x}, \mathbf{C}) + C_m$, is a special case of the function $V(\mathbf{x}, \alpha)$:

$$W(\mathbf{x}, \mathbf{C}) = (V + \alpha_q)|_{\lambda^{m-1}} \tag{2.2}$$

where $\lambda^{m-1} = (\alpha(\mathbf{C}), \alpha_q(\mathbf{C}))$ is a regular parameterized $(m-1)$-surface in the space of arbitrary constants α_j.

Let π denote a regular l-surface in the space of arbitrary constants α and α_q, passing through a fixed point $((\alpha_0, \alpha_{q0}))$, and let $K_l(\mathbf{x}_0, \alpha_0, \alpha_{q0})[V]$ be the set of envelopes (in the neighbourhood of $\mathbf{x}_0$) of all possible families $(V + \alpha_q)|_\pi$, where π runs through the whole set of regular l-surfaces that pass through the point (α_0, α_{q0}).

Definition [36]. The space

$$T[V] = \bigcup_{l=0}^{\min(q-1,n-1)} T_l[V], \qquad T_l[V] = \bigcup_{(\mathbf{x}_0,\alpha_0,\alpha_{q0})} K_l^{(\mathbf{x}_0,\alpha_0,\alpha_{q0})}[V] \tag{2.3}$$

will be called a functional extension of the solution space of Eqs (1.4), and $S = q - n$ will be called the degree of the space $T[V]$.

Obviously, $M \subseteq T[V]$.

It follows from this definition that $T[V]$ is the set of envelopes of all possible subfamilies of the functions $V + \alpha_q$, along with the set M, which admits of $W(\mathbf{x}, \mathbf{C}) + C_m$ as a generating function. In the classical case ($m - n \leqslant S \leqslant 0$) the definition of the set $T[V]$ is identical verbatim with the geometrical description of the solution space of a completely integrable system of first-order pde's whose complete integral is the function $V(\mathbf{x}, \alpha) + \alpha_0$ [33]. Hence, it follows that in the classical case $T[V]$ is simply the solution space of a certain system of equations of type (1.4). The number of equations in the system in $n - q + 1$.

Let us consider the non-classical case $S > 0$. Condition (2.2) guarantees the "structural compatibility" of M and $T[V]$[36]: $M \subseteq T[V]$, and moreover $M_l \subset T_l[V]$ $(l = 0, \ldots, h-1)$. Note that of all the subspaces $T_l[V]$, the simplest to construct is $T_0[V]$, since it consists of functions of the form $V(\mathbf{x}, \alpha) + \alpha_q$ $(\alpha, \alpha_q) = \text{const}$.

An important question in the theory of such spaces is their relation to differential equations. Here it is essential to prove theorems establishing a relation between the space $T[V]$ and the set of singular solutions of the linear Pfaft equation associated with the function V and covariant with respect to variation of the degree S; it is also important to formulate theorems establishing a relation between the space $T[V]$ and the space of all solutions of an integrable pde of higher order which is not covariant with respect to S.

Consider the Pfaff equation associated with V

$$\sum_{j=1}^{q-1} V'_{\alpha_j}(\mathbf{x},\alpha)d\alpha_j + d\alpha_q = 0 \qquad (q > n) \tag{2.4}$$

A solution vector $(\alpha(\mathbf{x}), \alpha_q(\mathbf{x}))$ of Eq. (2.4) is said to be singular if

$$\operatorname{rank} \partial(\boldsymbol{\alpha}, \alpha_q) / \partial\mathbf{x} < n \tag{2.5}$$

Theorem 1 [37]. A function $z(\mathbf{x})$ belongs to the space $T[V]$ if and only if a singular solution $(\alpha(\mathbf{x}), \alpha_q(\mathbf{x}))$ of Eq. (2.4) exists satisfying the condition

$$z(\mathbf{x}) = V(\mathbf{x}, \boldsymbol{\alpha}(\mathbf{x})) + \alpha_q(\mathbf{x}) \tag{2.6}$$

The classical version of Theorem 1 $(m - n \leqslant S \leqslant 0)$ is a theorem that provides a rigorous basis for the Lagrange method of variation of arbitrary constants as applied to first-order systems of pde's. Condition (2.5) is always satisfied in that case, since the differentials $d\alpha_j(\mathbf{x})$ are linearly dependent by virtue of Eq. (2.4).

Using (2.4) and (2.6), we will derive alternative forms of the degeneracy condition (2.5), which are independent of the vector $(\alpha(\mathbf{x}), \alpha_q(\mathbf{x}))$ and its derivatives with respect to $\mathbf{x}$. Let $S = 1$. Condition (2.5) has the form

$$\det \alpha_{\mathbf{x}} = 0 \tag{2.7}$$

since the differential $d\alpha_q$ is linearly dependent, by (2.4), on the differentials $d\alpha_j$ $(j = 1, \ldots, n)$.

Differentiating Eq. (2.6) with respect to $\mathbf{x}$, we obtain, by (2.4),

$$z_{\mathbf{x}} = V_{\mathbf{x}}(\mathbf{x}, \boldsymbol{\alpha}(\mathbf{x})) \tag{2.8}$$

Differentiating (2.8) with respect to $\mathbf{x}$, we get

$$(z - V)_{\mathbf{xx}} = V_{\boldsymbol{\alpha}\mathbf{x}} \cdot \boldsymbol{\alpha}_{\mathbf{x}}, \quad \det V_{\boldsymbol{\alpha}\mathbf{x}} \neq 0.$$

where the derivatives are evaluated with respect to the explicitly occurring variables. Hence, it follows that $\alpha_{\mathbf{x}}$ and $(z - V)_{\mathbf{xx}}$ are equivalent matrices. Therefore, condition (2.7) together with (2.8) becomes

$$\det(z(\mathbf{x}) - V(\mathbf{x}, \boldsymbol{\alpha}))_{\mathbf{xx}} = 0, \qquad z_{\mathbf{x}} = V_{\mathbf{x}}(\mathbf{x}, \boldsymbol{\alpha}) \tag{2.9}$$

Equation (2.9) is an n-dimensional analogue of the Monge–Ampère equation. In the special case $n = 2$ it is the classical Monge–Ampère equation

$$rt - s^2 = ar + 2bs + ct + \varphi, \qquad r = z_{x_1x_1}, \quad s = z_{x_1x_2}, \quad t = z_{x_2x_2} \tag{2.10}$$

whose coefficients a, b, c and φ are functions of x_1, x_2, z and of the partial derivatives z_{x1}, z_{x2}. In the case considered, these coefficients satisfy the additional condition

$$c(dx_1)^2 - 2bdx_1dx_2 + a(dx_2)^2 \equiv d^2V(\mathbf{x}, \boldsymbol{\alpha}(\mathbf{x}, z_{\mathbf{x}})), \qquad \varphi = b^2 - ac \tag{2.11}$$

and therefore Eq. (2.10) is of parabolic type.

For every function $z(\mathbf{x}) \in T[V]$, let us ignore those values of $\mathbf{x}$ for which the rank of the matrix $(z - V)_{\mathbf{xx}}$ is seen to decrease. Subject to this reservation, we have the following theorem.

Theorem 2 [37]. The space $T[V]$ of degree $S = 1$ is precisely the set of smooth solutions $z(\mathbf{x})$ of an equation of the Monge–Ampère type (2.9).

Now consider the case $S = 2$. An alternative form of (2.5) is

$$F(\mathbf{x}, z(\mathbf{x}), z_{\mathbf{x}}, z_{\mathbf{xx}}, z_{\mathbf{xxx}}) = 0 \tag{2.12}$$

Theorem 3.† The space $T[V]$ of degree $S = 2$ is precisely the set of smooth solutions $z(\mathbf{x})$ of some third-order equation (2.12).

The case $S \geqslant 3$ has not been investigated.

Note [37] that increasing the degree S by one yields an extension of the space $T[V]$ in such a way that it becomes a degenerate subset of its extension. This procedure induces inclusion of the equation corresponding to $T[V]$ among the intermediate integrals of the equation corresponding to the extended space. Thus, Eq. (1.5) is an intermediate integral of Eq. (2.9), Eq. (2.9) is an intermediate integral of Eq. (2.12), and so on.

Thus, in accordance with the heuristic requirement of the generalized approach, the required Lyapunov functions are elements of the space $T[V]$, that is, integrals of a certain integrable pde (or a system of pde's if $S < 0$).

In classical cases ($m - n \leqslant S \leqslant 0$), these spaces are functionally closed if the function $V(\mathbf{x}, \alpha)$ depends linearly on α_j: for any set of functions $T[V]$ and any smooth function $B(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, we have $B(G_1, \ldots, G_k) \in T[V]$.

However, if $S > 0$, the space $T[V]$ is no longer functionally closed, since it is described by a non-linear pde. Nevertheless, the following lemma establishes that these spaces are "partially functionally closed".

Lemma. Let $V(\mathbf{x}, \alpha) + \alpha_q(V(\mathbf{x}, \alpha) = \alpha_1 U_1(\mathbf{x}) + \ldots + \alpha_{q-1}U_{q-1}(\mathbf{x}), q > n)$ be a non-degenerate family of functions which depends linearly on the constants α_j, and let $\{G_1, \ldots, G_k\}$ $(k \leqslant n-1)$ be an arbitrary set of independent functions from the subspace $T_0[V]$. Then for any smooth function $B(y_1, \ldots, y_k)$,

$$B(G_1, \ldots, G_k) \in T[V]$$

Proof. It follows from the assumptions of the lemma that G_j may be expressed in the form

$$G_j = \alpha_1^{(j)} U_1 + \ldots + \alpha_{q-1}^{(j)} U_{q-1} + \alpha_q^{(j)}$$

where $\alpha_i^{(j)}$ are fixed values of the arbitrary constants α_i. Consider the non-degenerate function $W^* + C_{k+1}$, where

$$W^* = \sum_{j=1}^{k} C_j G_j = \sum_{i=1}^{q-1} U_i \left(\sum_{j=1}^{k} C_j \alpha_i^{(j)} \right) + \sum_{j=1}^{k} C_j \alpha_q^{(j)}, \quad C_j = \text{const} \tag{2.13}$$

It follows from this formula that $W^* + C_{k+1}$ is a special case of a function $V(\mathbf{x}, \alpha) + \alpha_q$, and therefore $T[W^*] \subset T[V]$. The space $T[W^*]$ is functionally closed, since $W^* + C_{k+1}$ is the complete integral of a linear homogeneous pde of type (1.5) if $k = n - 1$, or the complete integral of a system of linear equations of type (1.4) if $k < n - 1$. Since the family $\{G_1, \ldots, G_k\}$ belongs to the subset $T_0[W^*] \subset T[W^*]$, we have $B(G_1, \ldots, G_k) \in T[W^*] \subset T[V]$.

In the case $k = n$ the statement of the lemma is meaningless: the function $B(G_1, \ldots, G_n)$ does not necessarily belong to the space $T[V]$.

This lemma considerably simplifies the procedure of looking for v-functions in the space $T[V]$. For example, a sign definite function may be constructed using Chetayev's method of combinations of integrals, taking the integrals $G_1, \ldots, G_k$ to be solutions of pde's. Pozharitskii's theorem on the criteria for $B(G_1, \ldots, G_k)$ to be sign definite [38] remains valid, as do the results of investigations of $B(G_1, \ldots, G_k)$ based on the Hessian of this function [39].

Thus, let $V(\mathbf{x}, \alpha) + \alpha_q(V(\mathbf{x}, \alpha) = \alpha_1 U_1(\mathbf{x}) + \ldots + \alpha_{q-1}U_{q-1}(\mathbf{x}), q > m)$ be a non-degenerate family of functions which depends linearly on the constants α_j, a special case of which is a linear combination of

†KRASIL'NIKOV, P. S., Functional extensions in stability problems. Doctorate dissertation, 01.02.01, Moscow, 1996.

the integrals $W + C_m$, $W = C_1F_1 + \ldots + C_{m-1}F_{m-1}$ of the comparison system (1.2); let $T[V]$ be a functional extension of degree $S = q - n$ of the set

$$M = \{F(\mathbf{x}):\ F = B(F_1, \ldots, F_{m-1}),\quad B \in C^r(R^{m-1}, R^1)\}$$

The generalized scheme for constructing v-functions from first integrals is as follows. The required Lyapunov functions are sought over the set $T[V]$ in one of the following forms:

1) $v(\mathbf{x})$ is a linear function of the integrals $U_j(\mathbf{x})$ ($j = 1, \ldots, q-1$);

2) $v(\mathbf{x})$ is a non-linear function of independent integrals $U_j(\mathbf{x})$ ($j = 1, \ldots, k, k \leqslant n-1$); $v(\mathbf{x})$ may be constructed as a Chetayev combination of integrals:

$$v = \sum_{j=1}^{k} \lambda_j U_j + \sum_{j=1}^{k} \mu_j U_j^2, \quad \lambda_j, \mu_j = \text{const}$$

3) $v(\mathbf{x})$ is an element of the space $T[V]$ with the most general representation

$$v(\mathbf{x}) = \alpha_1(\mathbf{x})U_1(\mathbf{x}) + \ldots + \alpha_{q-1}(\mathbf{x})U_{q-1}(\mathbf{x}) + \alpha_q(\mathbf{x}) \tag{2.14}$$

where $((\alpha_1(\mathbf{x}), \ldots, \alpha_q(\mathbf{x}))$ is a solution vector of Eqs (2.4) satisfying the degeneracy condition (2.5).

4) $v(\mathbf{x})$ is the envelope of some l-parameter family $(V + \alpha_q)\pi$, where $\pi = (\alpha(C_1, \ldots, C_l), \alpha_q(C_1, \ldots, C_l))$ is a regular l-surface in the space of arbitrary constants α and α_q.

As shown above, the generalized scheme retains all the specific features of the classical approach, only augmenting it by having recourse to the solution set of higher-order equations.

One of the differences between the generalized scheme and the classical scheme is as follows. The construction of the required functions as non-linear functions of integrals $U_j(\mathbf{x})$ does not exhaust the entire set $T[V]$; this implies the need to use procedures 3 and 4 of the scheme in applied problems. The most general form of the function $v(\mathbf{x})$ as an element of the space $T[V]$ is represented by formula (2.14).

It was proved in the dissertation cited in footnote 2 that the majority of solved problems in stability theory in which v-functions were constructed in explicit form satisfy the generalized scheme.

3. NEW CRITERIA FOR ASYMPTOTIC STABILITY AT 1:1 RESONANCE

Let us consider the problem of the stability of an autonomous system of the form

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}),\quad \mathbf{X}(\mathbf{0}) = \mathbf{0},\quad \mathbf{x} \in \mathbf{R}^4 \tag{3.1}$$

where $\mathbf{X}(\mathbf{x})$ is a smooth vector field and the matrix $(\partial\mathbf{X}/\partial\mathbf{x})_0$ has pure imaginary eigenvalues λ_1 and λ_2 such that $\lambda_1 = \lambda_2$. Let us assume, moreover, that λ_1 has simple elementary divisors.

It has been shown [40] that this problem is algebraically unsolvable; the surface S separating the domain of asymptotic stability from the instability domain in the parameter space is transcendental. It has been shown [41] that this algebraic unsolvability is not "total": algebraic pieces of the surface have been constructed.

We will derive new and simpler criteria for asymptotic stability. The complex-valued normal form of the equations in the third approximation is

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 + A_{11} z_1^2 \bar{z}_1 + A_{12} z_1 z_2 \bar{z}_2 + A_1 z_1 \bar{z}_1 z_2 + A_2 z_1^2 \bar{z}_2 + A_3 z_2^2 \bar{z}_2 + A_4 z_2^2 \bar{z}_1 \\ \dot{z}_2 &= \lambda_1 z_2 + A_{21} z_1 \bar{z}_1 z_2 + A_{22} z_2^2 \bar{z}_2 + A_5 z_1^2 \bar{z}_1 + A_6 z_1 z_2 \bar{z}_2 + A_7 z_1^2 \bar{z}_2 + A_8 \bar{z}_1 z_2^2 \end{aligned} \tag{3.2}$$

where

$$z_1 = x_1 + ix_2,\quad z_2 = x_3 + ix_4,\quad A_{lm} = a_{lm} + ib_{lm},\quad A_m = a_m + ib_m$$

In polar coordinates r_j, θ_j associated with the variables z_j, $\bar{z}_j$ by the formulae

$$z_j = \sqrt{r_j}\exp(i\theta_j),\quad \bar{z}_j = \sqrt{r_j}\exp(-i\theta_j)$$

Eqs (3.2) have the following form (where $\theta = (\theta_1 - \theta_2)$ is the resonance angle)

$$\dot{r}_j = R_j(r_1, r_2, \theta), \quad \dot{\theta} = \Omega(r_1, r_2, \theta) \quad j = 1,2 \tag{3.3}$$

$$\tfrac{1}{2} R_1 = a_{11} r_1^2 + a_{12} r_1 r_2 + r_1 \sqrt{r_1 r_2} [(a_1 + a_2)\cos\theta + (b_2 - b_1)\sin\theta] +$$

$$+ r_2 \sqrt{r_1 r_2} (a_3 \cos\theta - b_3 \sin\theta) + r_1 r_2 (a_4 \cos 2\theta - b_4 \sin 2\theta)$$

$$\tfrac{1}{2} R_2 = a_{21} r_1 r_2 + a_{22} r_2^2 + r_2 \sqrt{r_1 r_2} [(a_6 + a_8)\cos\theta + (b_6 - b_8)\sin\theta] +$$

$$+ r_1 \sqrt{r_1 r_2} (a_5 \cos\theta + b_5 \sin\theta) + r_1 r_2 (a_7 \cos 2\theta + b_7 \sin 2\theta)$$

$$\Omega = (b_{21} - b_{11} + b_7 \cos 2\theta - a_7 \sin 2\theta) r_1 + (b_{22} - b_{12} - a_4 \sin 2\theta - b_4 \cos 2\theta) r_2 +$$

$$+ [(b_6 + b_8 - b_2 - b_1)\cos\theta + (a_8 - a_6 + a_2 - a_1)\sin\theta]\sqrt{r_1 r_2} + (b_5 \cos\theta - a_5 \sin\theta) r_1^{3/2} r_2^{-1/2} -$$

$$-(a_3 \sin\theta + b_3 \cos\theta) r_2^{3/2} r_1^{-1/2}$$

The required Lyapunov function will be constructed using the generalized scheme. As comparison equations, we choose the model system

$$\dot{r}_1 = \frac{\partial H}{\partial \theta_1}, \quad \dot{r}_2 = \frac{\partial H}{\partial \theta_2}, \quad \dot{\theta} = \frac{\partial H}{\partial r_1} - \frac{\partial H}{\partial r_2} \tag{3.4}$$

which has already been investigated [42] and is a special case of Eqs (3.3).

Here

$$H = \lambda_1 (r_1 + r_2) + \frac{1}{2}(b + b_{21} - b_{11}) r_1^2 + b r_1 r_2 + \frac{1}{2}(b - b_{22} + b_{12}) r_2^2 - 2(b_5 r_1 + b_3 r_2)\sqrt{r_1 r_2}\cos\theta +$$

$$+ 2(a_5 r_1 - a_3 r_2)\sqrt{r_1 r_2}\sin\theta - (b_4 \cos 2\theta + a_4 \sin 2\theta) r_1 r_2$$

is the normal form of the Hamiltonian function at multiple resonance. Equations (3.4) are integrable: the integral combination $W + C_3$, where

$$W = C_1 (H - \lambda_1 (r_1 + r_2)) + C_2 (r_1 + r_2)^2, \quad C_j = \text{const}$$

is a complete Lagrange integral of the linear homogeneous first-order pde corresponding to system (3.4). Consequently, the space of first integrals of Eqs (3.4) has the form

$$T[W] = T_0[W] \cup T_1[W] \cup T_2[W].$$

Consider the function

$$V = \alpha_1 r_1^2 + 2\alpha_2 r_1 r_2 + \alpha_3 r_2^2 + 2 r_1 \sqrt{r_1 r_2} (\alpha_4 \cos\theta + \alpha_5 \sin\theta) + 2 r_2 \sqrt{r_1 r_2} (\alpha_6 \cos\theta + \alpha_7 \sin\theta) +$$

$$+ 2 r_1 r_2 (\alpha_8 \cos 2\theta + \alpha_9 \sin 2\theta) \tag{3.5}$$

of which W is a special case. Here α_j are essential constants. The space $T[V]$ is an extension (of degree $S = 7$) of the whole set $T[W]$:

$$T[V] = T_0[V] \cup T_1[V] \cup T_2[V], \quad T_j[W] \subset T_j[V]$$

Let us search for auxiliary functions over the subset $T_2[V]$, which is filled out by the envelopes of all possible two-parameter families $(V + \alpha_{10})_\pi$, $\dim \pi = 2$. To that end, we consider a two-dimensional surface π in the space of arbitrary constants $\alpha_1, \ldots, \alpha_{10}$ of the function $(V + \alpha_{10})$,

$$\alpha_j = \gamma_{j1} v_1 + \gamma_{j2} v_2, \quad j = 1, \ldots, 9; \quad \alpha_{10} = (v_1^2 + v_2^2)/2$$

where v_1, and v_2 are local coordinates on that surface, and γ_{ij} are parameters. It is obvious that the envelope of the family $(V + \alpha_{10})\pi$ is

$$V = -\frac{1}{2}\sum_{k=1}^{2}[\gamma_{1k}r_1^2 + 2\gamma_{2k}r_1r_2 + \gamma_{3k}r_2^2 + 2r_1\sqrt{r_1r_2}(\gamma_{4k}\cos\theta + \gamma_{5k}\sin\theta) +$$

$$+2r_2\sqrt{r_1r_2}(\gamma_{6k}\cos\theta + \gamma_{7k}\sin\theta) + 2r_1r_2(\gamma_{8k}\cos 2\theta + \gamma_{9k}\sin 2\theta)]^2$$

Evaluating the derivative of V along the vector field of Eqs (3.3), we obtain

$$\dot{V} = r_2^5[\varkappa_0 + \sum_{m=1}^{5}(\varkappa_{m1}\cos m\theta + \varkappa_{m2}\sin m\theta)]$$

$$\varkappa_0 = G_5k^5 + G_4k^4 + G_3k^3 + G_2k^2 + G_1k + G_0$$

$$\varkappa_{1j} = \sqrt{k}(B_{4j}k^4 + B_{3j}k^3 + B_{2j}k^2 + B_{1j}k + B_{0j})$$

$$\varkappa_{2j} = k(D_{3j}k^3 + D_{2j}k^2 + D_{1j}k + D_{0j}), \quad \varkappa_{3j} = \sqrt{k}(L_{3j}k^3 + L_{2j}k^2 + L_{1j}k)$$

$$\varkappa_{4j} = k^2(M_{1j}k + M_{0j}), \quad \varkappa_{5j} = N_jk^{5/2}; \quad j = 1,2$$

where $k = r_1/r_2$ is a variable quantity and the coefficients $G_j, B_{ij}, D_{ij}, L_{ij}, M_{ij}, N_{ij}$, are quadratic functions of the quantity γ_{ij} and linear functions of the parameters of the problem. For example,

$$G_5 = 2\sum_{j=1}^{2}\gamma_{1j}[2a_{11}\gamma_{1j} + a_5\gamma_{4j} + b_5\gamma_{5j}], \quad G_0 = 2\sum_{j=1}^{2}\gamma_{3j}[2a_{22}\gamma_{3j} + a_3\gamma_{5j} - b_3\gamma_{7j}]$$

The expressions for the other coefficients are omitted. The constants γ_{ij} are chosen in such a way that the coefficients of the trigonometric functions of odd degrees vanish:

$$B_{ij} = L_{3j} = L_{2j} = L_{1j} = N_j = 0, \quad j = 1,2; \quad i = 0,\dots,4$$

We obtain an algebraic system of 18 non-linear equations of the form

$$\sum_{k=1}^{2}\sum_{i,j=1}^{9}R_{ij}^m\gamma_{ik}\gamma_{jk} = 0, \quad m = 1,\dots,18 \tag{3.6}$$

The coefficients R_{ij}^m are linear functions of the parameters a_{ij}, b_{ij}, a_j, b_j. The number of unknown quantities γ_{ik} is 18. The expressions for R_{ij}^m are omitted as they are very cumbersome.

We will show that system (3.6) has non-trivial solutions γ_{ik}. To that end, we will reduce Eqs (3.6) to a form in which the existence of a non-trivial family of solutions will be a corollary of the Implicit Function Theorem.

As Eqs (3.6) depend linearly on the parameters of the problem, they may be written as follows:

$$DA = 0 \tag{3.7}$$

$$A = (a_{11}, a_{12}, a_1 + a_2, b_2 - b_1, a_3, b_3, a_4, b_4, a_{21}, a_{22}, a_5, b_5, a_6 + a_8,$$

$$b_6 - b_8, a_7, b_7, b_{21} - b_{11}, b_{22} - b_{12}, b_6 + b_8 - b_2 - b_4, a_2 - a_1 - a_6 + a_8)^{\mathrm{T}}$$

where D is an 18×20 matrix whose elements d_{ij} are quadratic functions of γ_{ik} and A is the coefficient vector of system (3.3).

Calculations show that D is a non-singular matrix, since it has a non-zero minor of order 18 in its upper left corner. Hence it follows that the system is solvable for the first 18 elements of the vector A, and the dependence of these elements on the 18 parameters γ_{ik} of the function V is also non-degenerate. The unique feature of this representation of Eqs (3.6) is the inversion of the problem: the quantities γ_{ik} are chosen arbitrarily, while the parameters of system (3.3) are defined as solutions of Eqs (3.7).

Hence it follows, by the Implicit Function Theorem, that the dependence of the elements of A on the parameters γ_{ik} is locally solvable for γ_{ik} everywhere in the parameter space of the system (except for certain manifolds of zero measure).

Let γ_{ik}^* be a non-trivial solution of Eqs (3.6), depending on the parameters of Eqs (3.3). Let us consider the Lyapunov function V^* which is the restriction of V to this solution. The derivative of V^* is

$$\dot{V}^* = r_2^5[\varkappa_0^*(k) + \sum_{m=2,4}(\varkappa_{m1}^* \cos m\theta + \varkappa_{m2}^* \sin m\theta)] \tag{3.8}$$

Substituting $y = \mathrm{tg}(\theta/2)$ in the trigonometric polynomial on the right of (3.8), we have

$$\dot{V}^* = r_2^2\Lambda(y)(1+y^2)^{-2} \tag{3.9}$$

$$\Lambda(y) = L_4y^4 + L_3y^3 + L_2y^2 + L_1y + L_0$$

$$L_4 = \varkappa_0^* - \varkappa_{21}^* + \varkappa_{41}^*, \quad L_3 = 2(\varkappa_{22}^* - 2\varkappa_{42}^*), \quad L_2 = 2(\varkappa_0^* - 3\varkappa_{41}^*)$$

$$L_1 = 2(\varkappa_{22}^* + 2\varkappa_{42}^*), \quad L_0 = \varkappa_0^* + \varkappa_{21}^* + \varkappa_{41}^*$$

In the singular case $\theta = \pi$, when this substitution is degenerate, the function $\dot{V}^*$ may be evaluated using the formula

$$\dot{V}^* = r_2^2 L_4 \tag{3.10}$$

It will vanish if $L_4 = 0$, but then one of the roots of the polynomial $\Lambda(y)$ "departs" to infinity. It follows from (3.9) and (3.10) that the function $\dot{V}^*$ is sign-definite in the domain

$$r_1 \geqslant 0, \quad r_2 \geqslant 0, \quad 0 \leqslant \theta < \pi \tag{3.11}$$

if and only if, for any $k > 0$, the polynomial $\Lambda(y)$ has no real roots, including the point at infinity (in the planes $r_1 = 0, r_2 = 0$, the function $\dot{V}^*$ does not vanish, if one assumes that $G_0^* \neq 0$, $G_5^* \neq 0$).

It follows from the expression for the function V^* that it is always negative definite, with the exception of degenerate cases in which it is sign-definite (for an analytical description of these degenerate cases see [43]). In what follows we will disregard this case.

Theorem. Let $G_0^* \neq 0$, $G_5^* \neq 0$ and suppose that the real algebraic equation $\Lambda(y) = 0$ has no real roots for any $k > 0$, including the point at infinity. Then an equilibrium of the complete system is asymptotically stable if

$$\{\varkappa_0^* - \varkappa_{21}^* + \varkappa_{41}^*\} > 0 \tag{3.12}$$

and unstable otherwise.

Proof. It follows from the conditions of the theorem that the function $\dot{V}^*$ is sign-definite in the domain (3.11) (terms of higher order of smallness, which were omitted when deriving Eqs (3.2), do not affect the sign of $\dot{V}^*$, since the function V^* and the right-hand sides of Eqs (3.2) are homogeneous polynomials in $z_j, \bar{z}_j$). It follows from (3.9) that

$$\operatorname{sign}\dot{V}^* = \operatorname{sign}L_4 = \operatorname{sign}(\varkappa_0^* - \varkappa_{21}^* + \varkappa_{41}^*)$$

Let us consider the case in which inequality (3.12) holds. Since V^* is negative definite, it follows that $V^*\dot{V}^* < 0$, so that V^* satisfies all the conditions of Lyapunov's Asymptotic Stability Theorem.

If the inequality sign in condition (3.12) is reversed, the functions V^* and $\dot{V}^*$ will have the same signs in the neighbourhood of $r_1 = r_2 = 0$. Hence, by Lyapunov's Instability Theorem, the equilibrium position is unstable.

Thus, we have algebraic criteria for asymptotic stability in the domain $\dot{V}^* < 0$. Consequently, the transcendental surface S, separating the domain of asymptotic stability from the instability domain in the parameter space, is algebraic where $\dot{V}^* > 0$.

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